Math 247A Lecture 7 Notes

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1 Proofs of Interpolation Theorems

1.1 Proof of the Marcinkiewicz interpolation theorem

Last time, we introduced Hunt's interpolation theorem.

Theorem 1.1 (Hunt's interpolation theorem)**.** *Let $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ with $p_1 < p_2$ and $q_1 \neq q_2$. Assume that T is a sublinear map satisfying $\|Tf\|_{L^{q_j,\infty}} \lesssim \|f\|^*_{L^{p_j,1}}$ for $j = 1, 2$. Then, for any $1 \leq r \leq \infty$ and $\theta \in (0,1)$, we have*

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim \|f\|^*_{L^{p_\theta,r}}, \qquad \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Before proving this, we will prove the Marcinkiewicz interpolation theorem as a corollary.

Corollary 1.1 (Marcinkiewicz interpolation theorem)**.** *Let $1 \leq p_1 \leq q_1 \leq \infty$ and $1 \leq p_2 \leq q_2 \leq \infty$ with $p_1 \leq p_2$ and $q_1 \neq q_2$. Let T be a sublinear map that satisfies*

$$\|Tf\|^*_{L^{q_j,\infty}} \lesssim \|f\|_{L^{p_j}}, \qquad j = 1, 2.$$

Then for any $\theta \in (0,1)$, T is of strong type (p_θ, q_θ), where

$$\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Proof. As $p_1 \leq q_1$ and $p_2 \leq q_2$, we get $p_\theta \leq q_\theta$ for all $\theta \in (0,1)$. If $p_1 < p_2$, Hunt's theorem yields

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim \|f\|^*_{L^{p_\theta,r}} \qquad \forall 1 \leq r \leq \infty.$$

Taking $r = q_\theta$, we get

$$\|Tf\|_{L^{q_\theta}} \lesssim \|f\|^*_{L^{p_\theta,q_\theta}} \lesssim \|f\|_{L^{p_\theta}}.$$

Assume now that $p_1 = p_2 =: p$. Then

$$\|Tf\|^*_{L^{q_1,\infty}} \lesssim \|f\|_{L^p} \iff \sup_{\lambda > 0} \lambda |\{x : |Tf(x)| > \lambda\}|^{1/q_1} \lesssim \|f\|_p$$

$$\implies |\{x : |Tf(x)| > \lambda\}| \lesssim \left(\frac{\|f\|_p}{\lambda}\right)^{q_1} \qquad \forall \lambda > 0.$$

Similarly,

$$\|Tf\|^*_{L^{q_1,\infty}} \lesssim \|f\|_{L^p} \iff \sup_{\lambda>0} \lambda |\{x : |Tf(x)| > \lambda\}|^{1/q_2} \lesssim \|f\|_p$$
$$\implies |\{x : |Tf(x)| > \lambda\}| \lesssim \left(\frac{\|f\|_p}{\lambda}\right)^{q_2} \qquad \forall \lambda > 0.$$

We now have

$$\begin{aligned}\|Tf\|^{q_\theta}_{L^{q_\theta}} &= q_\theta \int_0^\infty \lambda^{q_\theta} |\{x : |(Tf)(x)| > \lambda\}| \,\frac{d\lambda}{\lambda} \\ &\lesssim \int_0^\infty \lambda^{q_\theta} \min\left\{\left(\frac{\|f\|_p}{\lambda}\right)^{q_1}, \left(\frac{\|f\|_p}{\lambda}\right)^{q_2}\right\} \frac{d\lambda}{\lambda}\end{aligned}$$

Say $q_1 < q_2$.

$$\begin{aligned}&\lesssim \int_0^{\|f\|_p} \lambda^{q_\theta} \left(\frac{\|f\|_p}{\lambda}\right)^{q_1} \frac{d\lambda}{\lambda} + \int_{\|f\|_p}^\infty \left(\frac{\|f\|_p}{\lambda}\right)^{q_2} \frac{d\lambda}{\lambda} \\ &\lesssim \|f\|_p^{q_1} \|f\|_p^{q_\theta - q_1} + \|f\|_p^{q_2} \|f\|_p^{q_\theta - q_2} \\ &\lesssim \|f\|_p^{q_\theta}.\end{aligned}$$

□

1.2 Proof of Hunt's interpolation theorem

Now let's prove Hunt's interpolation theorem. Recall that if $1 < p, q < \infty$, T is of restricted weak type (p, q) if

$$\|T\mathbb{1}_F\|^*_{L^{q,\infty}} \lesssim |F|^{1/p}$$

for every finite measure set F. We saw that this is equivalent to

$$\int |T\mathbb{1}_F(x)||\mathbb{1}_E(x)|\,dx \lesssim |F|^{1/p}|E|^{1/q'} \quad \forall E, F \iff ||Tf\|^*_{L^{q_\theta,\infty}} \lesssim \|f\|^*_{L^{p,1}} \quad \forall f \in L^{p,1}.$$

Proof. Claim: It suffices to prove Hunt for $1 < p_1, p_2, q_1, q_2 < \infty$. Indeed, for every $\theta \in (0, 1)$,

$$\|Tf\|^*_{L^{q_\theta,\infty}} \lesssim \|f\|^*_{L^{p_\theta,1}}$$

Indeed, for any $\theta \in (0, 1)$, even if $p_1 = 1$ and $q_1 = \infty$, $p_\theta \in (1, \infty)$. So we can use an interpolation argument with a slightly modified p_1 and p_2: It suffices to see that

$$\|T\mathbb{1}_F\|^*_{q_\theta,\infty} \lesssim |F|^{1/p_\theta}$$

for all finite measure sets F. We have

$$\begin{aligned}
\|T\mathbb{1}_F\|^*_{L^{q_\theta,\infty}} &= \sup_{\lambda>0} \lambda^{\theta+(1-\theta)}|\{x : |Tf(x)| > \lambda\}|^{\theta/q_1+(1-\theta)/q_2} \\
&\le \left(\sup_{\lambda>0} |\{x : |T\mathbb{1}_F(x)| > \lambda\}^{1/q_1}\right)^\theta \left(\sup_{\lambda>0} |\{x : |T\mathbb{1}_F(x)| > \lambda\}^{1/q_2}\right)^{1-\theta} \\
&= \left(\|T\mathbb{1}_F\|^*_{L^{q_1,\theta}}\right)^\theta \left(\|T\mathbb{1}_F\|^*_{L^{q_2,\infty}}\right)^{1-\theta} \\
&\lesssim |F|^{1/p_1\cdot\theta}|F|^{1/p_2\cdot(1-\theta)} \\
&\lesssim |F|^{1/p_\theta}.
\end{aligned}$$

Henceforth, we assume $1 < p_1, p_2, q_1, q_2 < \infty$. We can write

$$\|Tf\|^*_{L^{q_\theta,r}} \sim \sup_{\|g\|^*_{L^{q'_\theta,r'}}\le 1} \left|\int Tf(x)\cdot\overline{g(x)}\,dx\right|,$$

so it's enough to show that

$$\left|\int Tf(x)\overline{g(x)}\,dx\right| \lesssim 1 \qquad \forall \|f\|^*_{L^{p_\theta,1}} = 1, \|g\|^*_{L^{q'_\theta,r'}} \lesssim 1.$$

By splitting into real and imaginary parts (and then positive and negative parts), we may assume $f, g \ge 0$. We may also assume $g = \sum 2^m \mathbb{1}_{E_m}$, where E_m are measurable and pairwise disjoint. Caution: As T need not have monotonicity properties, we may not assume f is a simple function.

Using the binary expansion, we write

$$f(x) = \sum_{n\in\mathbb{Z}} 2^n a_n(x), \qquad a_n(x) \in \{0,1\}.$$

Note that there exists a largest $n(x)$ such that $a_{n(x)} = 1$ and $a_n(x) = 0$ for all $n > n(x)$. Also, we don't allow recurrent 1s. Let $\{n_k(x)\}_{k\ge1}$ be a decreasing sequence such that $a_{n_k(x)}(x) = 1$ and all other $a_n(x) = 0$. Then

$$f(x) = \sum_{k\ge1} 2^{n_k(x)}.$$

For $\ell \ge 1$, let $f_\ell(x) = 2^{n_\ell(x)}$. We can write

$$f_\ell(x) = \sum_{n\in\mathbb{Z}} 2^n \mathbb{1}_{F_n^\ell}, \qquad F_n^\ell = \{x : n_\ell(x) = n\}.$$

Then

$$f(x) = 2^{n_\ell(x)} + 2^{n_{\ell-1}(x)} + \cdots + 2^{n_1(x)}$$

$$
\begin{aligned}
&\geq 2^{n_\ell(x)} + 2^{n_\ell(x)+1} + \cdots + 2^{n_\ell(x)+\ell-1} \\
&= 2^{n_\ell(x)} \cdot (2^\ell - 1) \\
&\geq f_\ell(x) \cdot 2^{\ell-1}.
\end{aligned}
$$

So we get

$$f_\ell(x) \leq \frac{1}{2^{\ell-1}} f(x).$$

As $L^{p_\theta,r}$ is a Banach space, then $\sum_{\ell\geq 1} f_\ell = f$ in $L^{p_\theta,r}$.

Now we can tackle the bound:

$$
\begin{aligned}
\left|\int Tf(x)g(x)\,dx\right| &\leq \sum_{\ell\geq 1} \left|\int Tf_\ell(x) \left[\sum_m 2^m \mathbb{1}_{E_m}(x)\right] dx\right| \\
&\leq \sum_{\ell\geq 1} \sum_{n,m\in\mathbb{Z}} 2^n 2^m \int |T\mathbb{1}_{F_n^\ell}(x)||\mathbb{1}_{E_m}(x)|\,dx \\
&\lesssim \sum_{\ell\geq 1} \sum_{n,m} 2^n 2^m \min\{|F_n^\ell|^{1/p_1}|E_m|^{1/q_1'}, |F_n^\ell|^{1/p_2}|E_m|^{1/q_2'}\}.
\end{aligned}
$$

We will show that this is $\lesssim 1$ next time. $\square$